Example 4.3.4. Use the Leibniz rule and the chain rule to prove the quotient rule.

Proof. By the Leibniz rule, we have

$$\left(\frac{f}{g}\right)' = \left(f \cdot \frac{1}{g}\right)' = f' \cdot \frac{1}{g} + f \cdot \left(\frac{1}{g}\right)'.$$

For $\left(\frac{1}{g}\right)'$, let $y = \frac{1}{u}$ $= \frac{1}{g}$, where $u = g(x)$. Then, by the chain rule,

$$\left(\frac{1}{g}\right)' = \frac{dy}{du} \cdot \frac{du}{dx} = -\frac{1}{g^2(x)} g'(x).$$

$$(-u^{-2})\, g'$$

Therefore,

$$\left(\frac{f}{g}\right)' = f'\frac{1}{g} - f\frac{g'}{g^2} = \frac{f'g - fg'}{g^2}.$$

□

Example 4.3.5. Find

$$\frac{d}{dx} e^{\sqrt{x^2+x}}.$$

Solution.

$$\frac{dy}{dx} = e^{\sqrt{x^2+x}} \cdot (\sqrt{x^2+x})' \qquad \text{(using the chain rule; write} y = e^u, u = \sqrt{x^2+x}\text{)}$$

$$= e^{\sqrt{x^2+x}} \cdot \frac{1}{2}(x^2+x)^{-\frac{1}{2}} \cdot (2x+1) \quad \text{(using the chain rule again: let } u = \sqrt{w}, w = x^2+x\text{)}$$

■

Exercise 4.3.1. Prove that

1. $$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} g'(x).$$

2. $$\frac{d}{dx} e^{\sqrt{\frac{x-1}{x+1}}} = e^{\sqrt{\frac{x-1}{x+1}}} \cdot (x-1)^{-\frac{1}{2}} \cdot (x+1)^{-\frac{3}{2}}.$$

4.3.1 Some tricks involving the log function and its derivative

Example 4.3.6. Show that

$$\boxed{\frac{d}{dx}\ln|x| = \frac{1}{x}, \quad x \neq 0.}$$

Proof. Let

$$y = \ln|x| = \begin{cases} \ln x, & \text{if } x > 0 \\ \ln(-x), & \text{if } x < 0 \end{cases}$$

where $u = -x$

$y = \ln u$

For $x > 0$, $\dfrac{dy}{dx} = \dfrac{1}{x}$;

$\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx} = \dfrac{1}{u}(-1)$ $\quad \dfrac{du}{dx} = -1$

For $x < 0$, $\dfrac{dy}{dx} = \dfrac{1}{-x} \cdot (-1) = \dfrac{1}{x}$. (by the chain rule)

$= \dfrac{1}{-x}(-1)$ □

$= \dfrac{1}{x}$

Example 4.3.7. Let $y = \sqrt[3]{\dfrac{(x-2)(x-3)^2}{x-5}}$. Find $\dfrac{dy}{dx}$.

$\ln a^b = b \ln a$

$\ln(ab) = \ln a + \ln b$

Solution.

$$\begin{aligned} y^3 &= \frac{(x-2)(x-3)^2}{x-5} \\ \ln y^3 &= \ln\frac{(x-2)(x-3)^2}{x-5} \\ 3\ln y &= \ln(x-2) + 2\ln(x-3) - \ln(x-5) \\ \frac{3}{y}\frac{dy}{dx} &= \frac{1}{x-2} + \frac{2}{x-3} - \frac{1}{x-5} \\ \frac{dy}{dx} &= \frac{y}{3}\left(\frac{1}{x-2} + \frac{2}{x-3} - \frac{1}{x-5}\right) \\ \frac{dy}{dx} &= \frac{1}{3}\sqrt[3]{\frac{(x-2)(x-3)^2}{x-5}}\left(\frac{1}{x-2} + \frac{2}{x-3} - \frac{1}{x-5}\right) \end{aligned}$$

$\dfrac{d(3\ln y)}{dx}$

$= 3\dfrac{d}{dx}\ln y$

$= 3\dfrac{d}{dy}\ln y\dfrac{dy}{dx}$

$= \dfrac{3}{y}\dfrac{dy}{dx}$

$g = \ln(x-2)$ let $u = x-2$

$\dfrac{d\ln(x-2)}{dx} = \dfrac{d\ln u}{dx}$

$= \dfrac{d\ln u}{du}\dfrac{du}{dx}$

$= \dfrac{1}{u} \cdot 1 = \dfrac{1}{x-2}$

■

Remark. Alternatively, one may regard y as a function of x defined "implicitly" via the relation $(x-5)y^3 = (x-2)(x-3)^2$. (Cf. Chapter 5.)

Example 4.3.8. Compute the derivative of x^x, $\quad x > 0$.

$x = e^{\ln x}$

Solution. Write $x^x = e^{x \ln x}$. Let $y = e^u$, where $u = x \ln x$. Then

$y = x^x = (e^{\ln x})^x = e^{x \ln x}$

$= e^u$

can't use $(x^a)'$

nor $(a^x)'$

$$\frac{d}{dx}x^x = \frac{dy}{du}\frac{du}{dx}$$
$$= e^u(\ln x\frac{dx}{dx} + x\frac{d\ln x}{dx})$$
$$= e^u(\ln x + x\frac{1}{x})$$
$$= x^x(\ln x + 1).$$

$\frac{dy}{du} = e^u$

$\frac{du}{dx} = (x \ln x)'$

■

Exercise 4.3.2. Let $y = f(x)^{g(x)}$. Prove that $y' = f(x)^{g(x)}\left(g'(x)\ln f(x) + \frac{f'(x)}{f(x)}g(x)\right)$.

$\ln y = g \ln f.$

$\frac{d}{dx}\ln y = \frac{d}{dx}(g \ln f)$

$\frac{d \ln y}{dy}\frac{dy}{dx} \qquad \frac{dg}{dx}\ln f + g\frac{d}{dx}\ln f$

$\frac{1}{y}\frac{dy}{dx} \qquad = \frac{dg}{dx}\cdot \ln f + g\frac{df}{dx}\frac{d \ln f}{df}$

$= g' \ln f + g f' \cdot \frac{1}{f}$

$\frac{dy}{dx} = f^g \left(g' \ln f + g\frac{f'}{f}\right)$

Example 4.3.8. Compute the derivative of $x^x, \quad x > 0$.

Solution. Write $x^x = e^{x \ln x}$. Let $y = e^u$, where $u = x \ln x$. Then

$$\begin{aligned}
\frac{d}{dx}x^x &= \frac{dy}{du}\frac{du}{dx} \\
&= e^u(\ln x\frac{dx}{dx} + x\frac{d\ln x}{dx}) \\
&= e^u(\ln x + x\frac{1}{x}) \\
&= x^x(\ln x + 1).
\end{aligned}$$

■

Exercise 4.3.2. Let $y = f(x)^{g(x)}$. Prove that $y' = f(x)^{g(x)}\left(g'(x)\ln f(x) + \frac{f'(x)}{f(x)}g(x)\right)$.

MATH1520 University Mathematics for Applications — Fall 2021

Chapter 5: Differentiation II

Learning Objectives:
(1) Use implicit differentiation to find slope.
(2) Discuss inverse function and its derivatives.
(3) Study the higher order derivative.

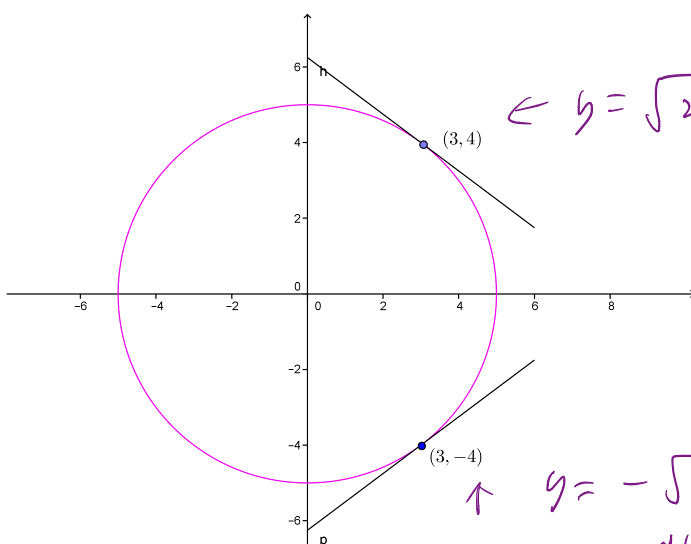
5.1 Differentiating Implicit Functions and Inverse Functions

5.1.1 Implicit functions

$C = \{(x, y) \mid x^2 + y^2 = 25\}$

Example 5.1.1. Consider the circle on the $x - y$ plane defined by $x^2 + y^2 = 25$. Find the equation of the tangent line to the circle at $(3, 4)$.

Solution. **Method 1. Express y in terms of x explicitly.**



$\leftarrow y = \sqrt{25 - x^2}$.

$\uparrow \; y = -\sqrt{25 - x^2}$

$\frac{dy}{dx} = \frac{d}{dx}\left(-\sqrt{25 - x^2}\right)$

$$x^2 + y^2 = 25 \quad \Rightarrow y = \pm\sqrt{25 - x^2},$$

Restrict to a small neighbourhood of the point $(3, 4)$ on the curve, $y > 0$ can be uniquely given by $y = \sqrt{25 - x^2}$.

So,

$$y' = -\frac{x}{\sqrt{25 - x^2}}$$

when $x = 3$, $y' = -\frac{3}{4}$. The equation of the tangent line to the curve at $(3, 4)$ is

at a point (x, y) on the tangent line

$$\frac{y-4}{x-3} = -\frac{3}{4}$$

$$4(y-4) = -3(x-3)$$

$$y - 4 = -\frac{3}{4}(x - 3),$$
$$y = -\frac{3}{4}x + \frac{25}{4}.$$

Method 2. Implicit differentiation.

Regard y as a function $y(x)$ without explicit formula. Differentiate both sides of $x^2+y^2 = 25$ with respect to x, and then solve algebraically for $\frac{dy}{dx}$.

$$\frac{dy^2}{dy}\frac{dy}{dx}$$

$$2x + \frac{d}{dx}(y^2) = 0$$
$$2x + 2y\frac{dy}{dx} = 0 \quad \text{(chain rule)}$$
$$\frac{dy}{dx} = -\frac{x}{y}$$

So,

$$\left.\frac{dy}{dx}\right|_{(3,4)} = -\frac{3}{4}.$$

Then, find the tangent line in the same way as with Method 1.

■

Remark. Method 2 is referred to as implicit differentiation, which is very useful to compute derivatives of functions not defined by explicit formulae.

Example 5.1.2. Let $y = f(x)$ be a differentiable function of x that satisfies the equation $x^2y + y^2 = x^3$. Find the derivative $\dfrac{dy}{dx}$ as a function of both x and y.

Solution. You are going to differentiate both sides of the given equation with respect to x. So that you will not forget that y is actually a function of x, temporarily use the alternative notation $f(x)$ for y, and begin by rewriting the equation as

$$x^2 f(x) + (f(x))^2 = x^3.$$

$$\frac{d}{dx}(x^2y + y^2) = \frac{d}{dx}x^3 = 3x^2.$$

$$= \frac{dx^2}{dx}y + x^2\frac{dy}{dx} + \frac{dy^2}{dx} = 2x \cdot y + x^2\frac{dy}{dx} + \underbrace{\frac{dy^2}{dy}}_{2y}\frac{dy}{dx}$$

Now differentiate both sides of this equation term by term with respect to x:

$$\begin{aligned} \frac{d}{dx}[x^2 f(x) + (f(x))^2] &= \frac{d}{dx}[x^3] \\ \rightsquigarrow \left[x^2 \frac{df}{dx} + f(x)\frac{d}{dx}(x^2)\right] + 2f(x)\frac{df}{dx} &= 3x^2. \end{aligned} \tag{5.1}$$

Thus, we have

$$\begin{aligned} x^2 \frac{df}{dx} + f(x)(2x) + 2f(x)\frac{df}{dx} &= 3x^2 \\ \rightsquigarrow [x^2 + 2f(x)]\frac{df}{dx} &= 3x^2 - 2xf(x) \\ \rightsquigarrow \frac{dy}{dx} &= \frac{3x^2 - 2xf(x)}{x^2 + 2f(x)}. \end{aligned} \tag{5.2}$$

Finally, replace $f(x)$ by y to get

$$\frac{dy}{dx} = \frac{3x^2 - 2xy}{x^2 + 2y}.$$

■

Remark. By default, $\frac{dy}{dx}$ is regarded as a function of x, and we want an expression for $\frac{dy}{dx}$ in terms of x only. However, sometimes it is difficult to express y in terms of x explicitly. In this case it'll be specified in the test or homework question that it is ok to leave the answer for y' as a function of both x and y. Or, sometimes finding the value for y' is only an intermediate step in solving the problem. If the values of x and y are known, one may directly plug in these values to the expression of y' in x and y, without going through an explicit formula for y' in x.

Summary: Carrying out Implicit Differentiation

Suppose an equation defines y implicitly as a differentiable function of x. To find $\dfrac{dy}{dx}$:

1. Differentiate both sides of the equation with respect to x. Remember that y is really a function of x, and use the chain rule when differentiating terms containing y.

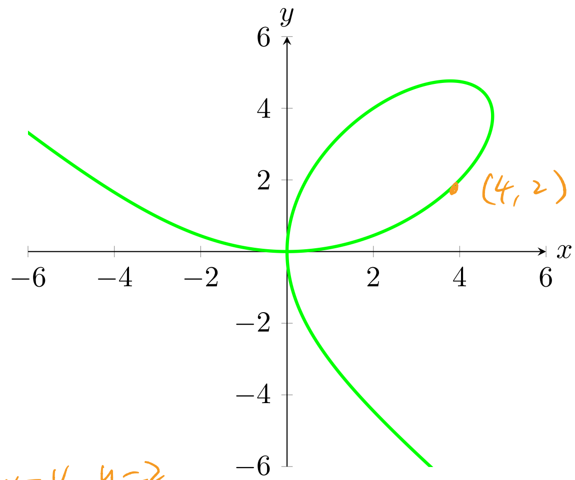
2. Solve the differentiated equation algebraically for $\dfrac{dy}{dx}$ in terms of x and y.

Example 5.1.3. Consider the curve defined by

$$x^3 + y^3 = 9xy.$$

1. Compute $\dfrac{dy}{dx}$. (It is ok to leave the answer as a function of both x and y.)

2. Find the slope of the tangent line to the curve at $(4, 2)$.



x=4, y=2 $\quad$ 64+8 = 9·4·2

Figure 5.1: A plot of $x^3 + y^3 = 9xy$. While this is not a function of y in terms of x, the equation still defines a relation between x and y.

Solution. Starting with

$$x^3 + y^3 = 9xy,$$

we apply the differential operator $\frac{d}{dx}$ to both sides of the equation to obtain

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx} 9xy.$$

Applying the sum rule, we see that

$$\frac{d}{dx}x^3 + \frac{d}{dx}y^3 = \frac{d}{dx}9xy.$$

Let's examine each of the terms above in turn. To begin,

$$\frac{d}{dx}x^3 = 3x^2.$$

On the other hand, $\frac{d}{dx}y^3$ is treated somewhat differently. Here, viewing $y = y(x)$ as an implicit function of x, we have by the chain rule that

$$\begin{aligned}\frac{d}{dx}y^3 &= \frac{d}{dx}(y(x))^3 \\ &= 3(y(x))^2 \cdot y'(x) \\ &= 3y^2\frac{dy}{dx}.\end{aligned}$$

$\frac{dy^3}{dy}\frac{dy}{dx} = 3y^2\frac{dy}{dx}$

Consider the final term $\frac{d}{dx}(9xy)$. Regarding $y = y(x)$ again as an implicit function, we have:

$$\begin{aligned}\frac{d}{dx}(9xy) &= 9\frac{d}{dx}\left(x \cdot y(x)\right) \\ &= 9\left(x \cdot y'(x) + y(x)\right) \\ &= 9x\frac{dy}{dx} + 9y.\end{aligned}$$

Putting all the above together, we get:

$$3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y.$$

Now we solve the preceding equation for $\frac{dy}{dx}$. Write

$$\begin{aligned} & 3x^2 + 3y^2\frac{dy}{dx} = 9x\frac{dy}{dx} + 9y \\ \iff \quad & 3y^2\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2 \\ \iff \quad & \frac{dy}{dx}\left(3y^2 - 9x\right) = 9y - 3x^2 \\ \iff \quad & \frac{dy}{dx} = \frac{9y - 3x^2}{3y^2 - 9x} = \frac{3y - x^2}{y^2 - 3x}.\end{aligned}$$

For the second part of the problem, we simply plug in $x = 4$ and $y = 2$ to the last formula above to conclude that the slope of the tangent line to the curve at $(4, 2)$ is $\frac{5}{4}$. See Figure 5.2. ■

Example 5.1.4. Let L be the curve in the $x - y$ plane defined by $x^2 + y^2 + e^{xy} = 2$. Use L to implicitly define a function $y = y(x)$. Find $y'(x)$ at $x = 1$ and the tangent line to the curve L at $(1, 0)$.

Solution. (Note: In this case, there is no good explicit formula for the function $y(x)$.) Differentiate the equation $x^2 + y^2 + e^{xy} = 2$ on both sides with respect to x. We get:

$$\begin{aligned} 2x + 2yy' + e^{xy}(y + xy') &= 0, \\ \rightsquigarrow y' &= -\frac{2x + e^{xy}y}{2y + e^{xy}x}.\end{aligned}$$

$$\frac{dy^2}{dx} = \frac{dy^2}{dy}\frac{dy}{dx} = 2y\frac{dy}{dx}$$

So, $y(1) = 0$ and $y'|_{x=1)} = -2$.

where $u = xy$

$$\frac{de^{xy}}{dx} = \frac{de^u}{du}\frac{du}{dx} = e^u(x'y + xy') = e^{xy}(y + xy')$$

$$(2y + xe^{xy})y' = -2x - e^{xy}y$$

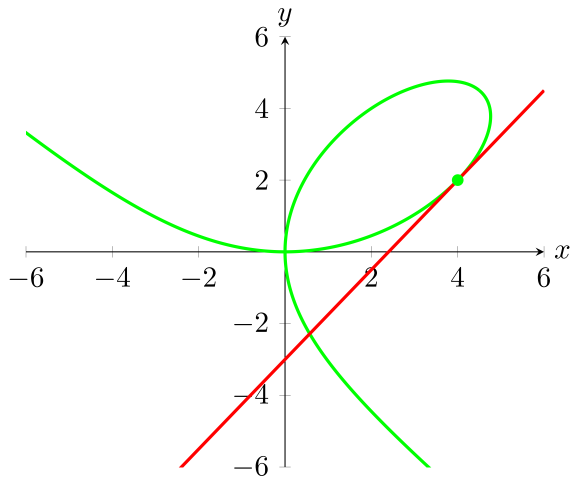


Figure 5.2: A plot of $x^3 + y^3 = 9xy$ along with the tangent line at $(4, 2)$.

Thus, the equation of the tangent line to L at $(x, y) = (1, 0)$ is:

$$y - 0 = -2(x - 1), \quad \text{or}$$
$$y = -2x + 2.$$

■

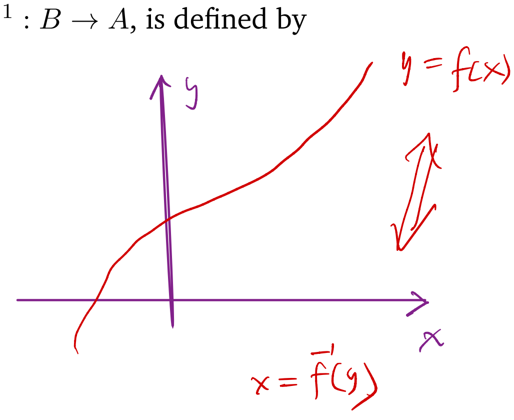
5.1.2 Differentiating Inverse Functions

Definition 5.1.1. Consider a function $f : A \to B$, where A is the domain, and B is the codomain.

The function f is said to be *injective* if $f(x_1) \neq f(x_2)$ whenever $x_1 \neq x_2$ for any $x_1, x_2 \in A$. The function f is said to be *surjective* or *onto* if $\forall y \in B$, $\exists x \in A$ such that $f(x) = y$. (In this case, the codomain B of f agrees with the range of f.) The function f is said to be *bijective* or *one to one* if it is both injective and surjective.

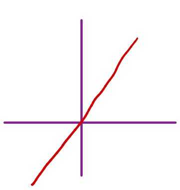
If f is one-to-one, then the *inverse function*, denoted $f^{-1} : B \to A$, is defined by

$$x = f^{-1}(y) \quad \text{if } y = f(x).$$

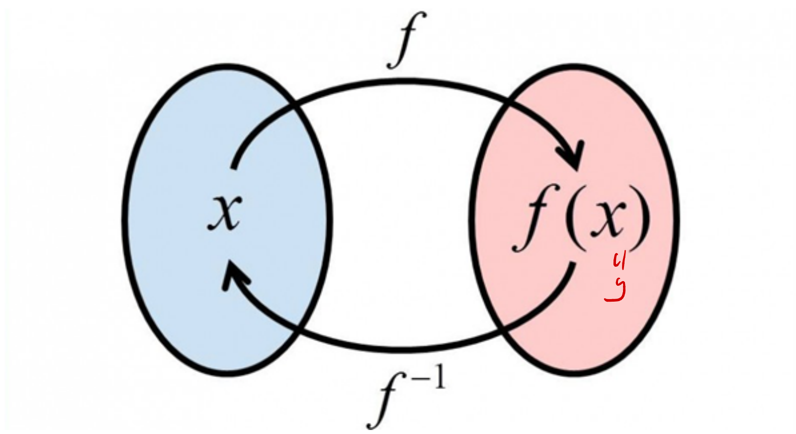


Remark.

1. Only a one-to-one function can have an inverse.



E.g., $f(x) = x$ is one-to-one
$f^{-1}(x) = x$



2. The domains and codomains(=ranges) of f and f^{-1} are interchanged.
3. $f^{-1}(x)$ is **not** $\dfrac{1}{f(x)}$. $= (f(x))^{-1} = f(x)^{-1}$
4.

$$\begin{aligned} (f^{-1} \circ f)(x) &= x, && \text{for all } x \text{ in the domain of } f \\ (f \circ f^{-1})(y) &= y, && \text{for all } y \text{ in the domain of } f^{-1} \text{ (or range of } f) \end{aligned}$$

Example 5.1.5.

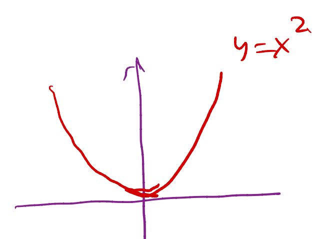
1. $$\begin{cases} y = e^x, \\ x = \ln y. \end{cases} \qquad x \in \mathbb{R}, y > 0$$

   $e^{\ln x} = x$

   $\ln e^x = x$

   are inverse functions of each other.

2. $$\begin{cases} y = x^2, \\ x = \sqrt{y}. \end{cases} \qquad x > 0, y > 0$$

   are inverse functions of each other.

3. $y = x^2$, $x \in \mathbb{R}, y \geq 0$ does not have inverse function because it is not one-to-one.

Question: What is the relation between derivatives of inverse functions?

Suppose $y = f(x)$ has an inverse function, then

$$x = f^{-1}(f(x)).$$

$y = f(x)$ then $x = f^{-1}(y)$

Differentiate both sides with respect to x to get:

$$1 = (f^{-1})'(y) \cdot f'(x)$$

$$\iff \quad \boxed{(f^{-1})'(y) = \frac{1}{f'(x)},}$$

$f^{-1}(f(x)) = x$

$\frac{d f^{-1}(f(x))}{dx} = \frac{dx}{dx} = 1$

$= \frac{d f^{-1}(y)}{dx} = \frac{d f^{-1}(y)}{dy} \frac{dy}{dx}$

or equivalently,

$$\boxed{\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}.}$$

$\frac{dx}{dy} = \frac{d f^{-1}(y)}{dy} = \frac{1}{\frac{dy}{dx}}$

Example 5.1.6. Use the identity $\frac{d}{dx}e^x = e^x$ to show that

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Solution. Let $y = f(x) = \ln x$. Then its inverse function is $x = e^y$.

$$\frac{dy}{dx} = \frac{d}{dx}\ln x = \frac{1}{\frac{dx}{dy}} = \frac{1}{e^y}.$$

$x = e^y \quad \frac{dx}{dy} = e^y$

Express the right hand side in terms of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}.$$

Or, using implicit differentiation: Differentiate the equation $x = e^y$ on both sides with respect to x. We get:

$$1 = \frac{d}{dx}(e^y) \overset{\frac{de^y}{dy}\frac{dy}{dx}}{=} e^y \cdot \frac{dy}{dx} \qquad \text{(the chain rule)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{d}{dx}\ln x = \frac{1}{e^y} = \frac{1}{x}.$$

■

Example 5.1.7. Show that

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Solution. Let $y = \sqrt{x}$, then $x = y^2$. We have:

$$\frac{d\sqrt{x}}{dx} = \frac{dy}{dx} = \left(\frac{dx}{dy}\right)^{-1} = \frac{1}{2y}.$$

Expressing the right hand side in terms of x, we have

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}.$$

■

Example 5.1.8. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = x^3 + 4x$.

$f(y) = y^3 + 4y$
$f' = 3y^2 + 4$

1. Find $\frac{d}{dx} f^{-1}(x)$ without writing down an explicit formula for $f^{-1}(x)$.
2. Find $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=5}$.

Solution.

1. Let $y = f^{-1}(x)$, i.e., $x = f(y)$. Then

$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{3y^2 + 4}.$$

Alternatively, differentiate both sides of the equation $x = y^3 + 4y$ with respect to x, regarding x now as an implicit function of y. We get:

$$\frac{dx}{dy} = 3y^2 + 4 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{3y^2 + 4}.$$

2. When $x = 5$, $y = f^{-1}(5) = 1$. (Check that $f(1) = 5$!) So,

$$\left.\frac{d}{dx} f^{-1}(x)\right|_{x=5} = \left.\frac{1}{3y^2 + 4}\right|_{y=1} = \frac{1}{7}.$$

$x = f(y) = y^3 + 4y$

■

5.2 Higher Order Derivatives

Suppose that an object is moving along a coordinate line, and let t denote the time. parametrized by t. Let

$$s = s(t)$$

denote the coordinate of the object at time t. The *velocity* (or "instantaneous velocity") of the object at time t is:

$$v(t) = s'(t).$$

The *acceleration* of the object at time t is:

$$a(t) = v'(t) = s''(t).$$

Notation Let $y = f(x)$.

1st derivative of f: $\dfrac{dy}{dx} = \dfrac{df}{dx} = f'(x)$

2nd derivative of f: $\dfrac{d^2y}{dx^2} = \dfrac{d^2f}{dx^2} = f''(x)$ $= \frac{1}{dx}\left(\frac{df}{dx}\right)$

⋮ ⋮

n-th derivative of f: $\dfrac{d^ny}{dx^n} = \dfrac{d^nf}{dx^n} = f^{(n)}(x)$

Example 5.2.1. $\frac{d}{dx}\frac{d}{dx}(a^x) = \frac{d a^x}{dx}(\ln a) = a^x(\ln a)^2$

1. $$\frac{d^n}{dx^n}(e^x) = e^x, \quad \frac{d^n}{dx^n}(a^x) = a^x \cdot (\ln a)^n.$$

2. $y = x^n, n \in \mathbb{N}$.

$$y^{(m)} = \begin{cases} n(n-1)(n-2)\cdots(n-m+1)x^{n-m}, & \text{if } m < n, \\ n(n-1)(n-2)\cdots 2\cdot 1 = n!, & \text{if } m = n, \\ 0, & \text{if } m > n. \end{cases}$$

Example 5.2.2. Let y be defined implicitly by the equation $x^2 + y^2 + e^{xy} = 2$. Find y' and y'' at $x = 1$.

Solution. Differentiate both sides of the preceding equation with respect to x to get

$$2x + 2yy' + e^{xy}(y + xy') = 0. \quad ----(1)$$

Then differentiate both sides of the equation with respect to x one more time to get

$$2 + 2y'y' + 2yy'' + e^{xy}(y + xy')^2 + e^{xy}(2y' + xy'') = 0. \quad ----(2)$$